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Quasifinite representations of a Lie algebra of Block type

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Abstract

Let $\mathcal{B}$ be the Lie algebra of Block type over $\mathbb{C}$ with basis $\{L_{\alpha,i}, C \mid \alpha, i \in \mathbb{Z}, i \geqslant -1\}$ and relations $[L_{\alpha,i}, L_{\beta,j}] = ((i+1)\beta - (j+1)\alpha)L_{\alpha+\beta,i+j} + \alpha\delta_{\alpha+\beta,0}\delta_{i+j,-2}C$, $[C, L_{\alpha,i}] = 0$. In this paper, it is proved that a quasifinite irreducible $\mathcal{B}$-module is a highest or lowest weight module. Furthermore, the quasifinite irreducible highest weight modules are classified and the unitary ones are proved to be trivial.

1. Introduction

Block [1] introduced a class of infinite dimensional simple Lie algebras over a field of characteristic zero. Generalizations of Block algebras (usually referred to as *Lie algebras of Block type*) have been studied by many authors (see, for example, [5,13,14,17–23]). Partially because they are closely related to the Virasoro algebra (and some of them are sometimes called Virasoro-like algebras [21]), these algebras have attracted some attention in the literature. The structure theory of these algebras has been developed, however, their representation theory does not seem to be well-developed yet.

In this paper, we study representations of the universal central extension $\mathcal{B} = \overline{\mathcal{B}} \oplus \mathbb{C}C$ of the Lie algebra of Block type, where $\overline{\mathcal{B}} = \mathrm{span}\{L_{\alpha,i} \mid \alpha, i \in \mathbb{Z}, i \geqslant -1\}$ with the bracket $[L_{\alpha,i}, L_{\beta,j}] = ((i+1)\beta - (j+1)\alpha)L_{\alpha+\beta,i+j}$. The central extension $\mathcal{B}$ is defined by the

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2-cocycle ψ on $\overline{\mathcal{B}}$: $\psi(L_{\alpha,i}, L_{\beta,j}) = \alpha\delta_{\alpha+\beta,0}\delta_{i+j,-2}$ (its uniqueness can be proved as in [5]) and thus has the following bracket:

$$[L_{\alpha,i}, L_{\beta,j}] = \big((i+1)\beta - (j+1)\alpha\big)L_{\alpha+\beta,i+j} + \alpha\delta_{\alpha+\beta,0}\delta_{i+j,-2}C, \tag{1.1}$$

for $\alpha, \beta \in \mathbb{Z}$, $i, j \geqslant -1$, where C is a central element.

We observe a deep fact that the Lie algebra $\mathcal{B}$ can be related to the well-known Lie algebra $W_{1+\infty}$ of $\mathcal{W}$-infinity algebras in the following way: $W_{1+\infty} = \mathbb{C}[x, x^{-1}, \mathrm{d}/\mathrm{d}x] \oplus \mathbb{C}C$ is the universal central extension of the Lie algebra of differential operators on the circle with bracket

$$\big[x^\alpha D^i, x^\beta D^j\big] = x^{\alpha+\beta}\big((D+\beta)^i D^j - D^i(D+\alpha)^j\big) + \phi\big(x^\alpha D^i, x^\beta D^j\big)C,$$

for $\alpha, \beta \in \mathbb{Z}$, $i, j \geqslant 0$, where $D = x(\mathrm{d}/\mathrm{d}x)$ and $\binom{i}{j}$ denotes the binomial coefficient, and where ϕ is a 2-cocycle defined by

$$\phi\left(x^{\alpha+i}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i, x^{\beta+j}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j\right) = \delta_{\alpha+\beta,0}(-1)^i i!\,j!\binom{\alpha+i}{i+j+1}.$$

Define a natural filtration of $W_{1+\infty}$ by

$$\{0\} = (W_{1+\infty})_{[-3]} \subset (W_{1+\infty})_{[-2]} \subset \cdots \subset W_{1+\infty},$$

where

$$(W_{1+\infty})_{[n]} = \operatorname{span}\big\{x^\alpha D^i, C \mid \alpha \in \mathbb{Z},\ i \leqslant n+1\big\} \quad \text{for } n \geqslant -2.$$

Then $\mathcal{B}$ is simply the associated graded Lie algebra of the filtered Lie algebra $W_{1+\infty}$.

The $\mathcal{W}$-infinity algebras arise naturally in various physical theories, such as conformal field theory, the theory of the quantum Hall effect, etc.; among them the most fundamental one is the $W_{1+\infty}$ algebra, whose representation theory, of interest to both mathematicians and physicists, has been well-developed (see, for example, [2,3,6–10,15]).

We can realize the Lie algebra $\mathcal{B}$ in the space $\mathbb{C}[x, x^{-1}, t] \oplus \mathbb{C}C$ with the bracket

$$\big[x^\alpha f(t), x^\beta g(t)\big] = x^{\alpha+\beta}\big(\beta f'(t)g(t) - \alpha f(t)g'(t)\big) + \alpha\delta_{\alpha+\beta,0}f(0)g(0)C, \tag{1.2}$$

for $\alpha, \beta \in \mathbb{Z}$, $f(t), g(t) \in \mathbb{C}[t]$, where the prime stands for $\mathrm{d}/\mathrm{d}t$. Then $\mathcal{B}$ has a natural $\mathbb{Z}$-gradation: $\mathcal{B} = \bigoplus_{\alpha\in\mathbb{Z}} \mathcal{B}_\alpha$ with

$$\mathcal{B}_\alpha = \big\{t^\alpha f(t) \mid f(t) \in \mathbb{C}[t]\big\} + \delta_{\alpha,0}\mathbb{C}C. \tag{1.3}$$

When we study representations of a Lie algebra of this kind, as pointed in [7,8], we encounter the difficulty that though it is $\mathbb{Z}$-graded, the graded subspaces are still infinite dimensional, thus the study of quasifinite modules is a nontrivial problem.

The paper is organized as follows. In Section 2, after introducing a family of Lie algebras $\mathcal{B}(\Gamma)$ of Block type, we prove that $\mathcal{B}(\Gamma)$ has a nontrivial quasifinite module if and only if $\Gamma \cong \mathbb{Z}$ and that a quasifinite irreducible module over $\mathcal{B} = \mathcal{B}(\mathbb{Z})$ is a highest or lowest weight module (Theorem 2.1). Then in Sections 3 and 4, we classify quasifinite irreducible highest weight modules and furthermore, the unitary ones are proved to be trivial (Theorems 3.4 and 4.3).

2. Classification theorem

Let us start with general settings. Let $\mathbb{F}$ be a field of characteristic zero, Γ a nonzero additive subgroup of $\mathbb{F}$. Then there is a *Lie algebra of Block type* $\mathcal{B}(\Gamma)$ (see, for example, [1,19,22]) defined on the space $\mathbb{F}[\Gamma] \otimes \mathbb{F}[t] \oplus \mathbb{F}C$ with bracket (1.2) for $\alpha, \beta \in \Gamma$ and $f(t), g(t) \in \mathbb{F}[t]$, where $\{x^\alpha \mid \alpha \in \Gamma\}$ is the standard basis of the group algebra $\mathbb{F}[\Gamma]$.

Choose a total ordering of Γ compatible with its group structure (when $\Gamma = \mathbb{Z}$, we always choose the natural ordering, and in this case, $\mathcal{B} = \mathcal{B}(\mathbb{Z})$). Then $\mathcal{B}(\Gamma) = \bigoplus_{\alpha \in \Gamma} \mathcal{B}(\Gamma)_\alpha$ is Γ-graded with graded subspaces defined as in (1.3). Setting $\mathcal{B}(\Gamma)_\pm = \bigoplus_{\pm\alpha > 0} \mathcal{B}(\Gamma)_\alpha$, we have the triangular decomposition

$$\mathcal{B}(\Gamma) = \mathcal{B}(\Gamma)_- \oplus \mathcal{B}(\Gamma)_0 \oplus \mathcal{B}(\Gamma)_+.$$

Note that $\mathcal{B}(\Gamma)_0 = \mathbb{F}[t] + \mathbb{F}C$ is a commutative subalgebra of $\mathcal{B}(\Gamma)$ (but it is not a Cartan subalgebra).

A module V over $\mathcal{B}(\Gamma)$ is *Γ-graded* if $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ and $\mathcal{B}(\Gamma)_\alpha V_\beta \subset V_{\alpha+\beta}$ for all α, β; *quasifinite* if $\dim V_\beta < \infty$ for all β; *uniformly bounded* if there is $N > 0$ such that $\dim V_\beta \leqslant N$ for all β; a *module of the intermediate series* if $\dim V_\beta \leqslant 1$ for all β.

Given $\Lambda \in \mathcal{B}(\Gamma)_0^*$ (the dual space of $\mathcal{B}(\Gamma)_0$), a *highest* (respectively *lowest*) *weight module* over $\mathcal{B}(\Gamma)$ is a Γ-graded module $V(\Lambda)$ generated by a *highest* (respectively *lowest*) *weight vector* $v_\Lambda \in V(\Lambda)_0$, i.e., v_Λ satisfies

$$h v_\Lambda = \Lambda(h) v_\Lambda \quad \text{where } h \in \mathcal{B}(\Gamma)_0, \quad \text{and}$$
$$\mathcal{B}(\Gamma)_+ v_\Lambda = 0 \quad \big(\text{respectively } \mathcal{B}(\Gamma)_- v_\Lambda = 0\big).$$

A nonzero vector v in a Γ-graded module V is called *singular* or *primitive* if $\mathcal{B}(\Gamma)_+ v = 0$.

Our first result is the following theorem.

Theorem 2.1. *A quasifinite irreducible $\mathcal{B}$-module is a highest or lowest module.*

The analogous results for affine Lie algebras, the Virasoro algebra, higher rank Virasoro algebras, and Lie algebras of Weyl type were obtained in [4,11,15,16] (in our case here, there does not exist a nontrivial module of the intermediate series, nor a nontrivial uniformly bounded module).

The proof of the above theorem will follow from Lemmas 2.2–2.3 below.

We denote

$$L_{\alpha,i} = x^\alpha t^{i+1} \quad \text{for } \alpha \in \Gamma,\ i \geqslant -1.$$

Then (1.2) is equivalent to (1.1). Denote

$$\mathrm{Vir}(\Gamma) = \mathrm{span}\{L_{\alpha,0} \mid \alpha \in \Gamma\}.$$

Then $\mathrm{Vir}(\Gamma)$ forms a centerless (generalized) Virasoro algebra under bracket (1.1) (cf. [15]).

Lemma 2.2. *Suppose V is a quasifinite irreducible $\mathcal{B}$-module without highest and lowest weights. Then*

$$L_{\pm 1,-1} : V_\alpha \to V_{\alpha \pm 1},$$

are injective, thus bijective for $\alpha \in \mathbb{Z}$. In particular, V is uniformly bounded.

Proof. Say $L_{1,-1}v_0 = 0$ for some $0 \neq v_0 \in V_0$. Since $C|_{V_0}, L_{0,-1}|_{V_0}, L_{0,0}|_{V_0}, \ldots$ are linear transformations on the finite-dimensional vector space V_0, there exists $s \geqslant 2$ such that for all $k \geqslant s$, $L_{0,k}|_{V_0}$ are linear combinations of $C|_{V_0}, L_{0,p}|_{V_0}$ for $0 \leqslant p < s$. This implies that

$$\mathcal{B}_0 v_0 = S v_0, \quad \text{where } S = \mathrm{span}\{C, L_{0,p} \mid 0 \leqslant p < s\}.$$

Notice that the adjoint operator $\mathrm{ad}_{L_{1,-1}}$ is locally nilpotent such that $\mathcal{B}_\alpha = \mathrm{ad}^\alpha_{L_{1,-1}}(\mathcal{B}_0)$ for $\alpha > 0$. Choose $\beta > 0$ such that $\mathrm{ad}^\beta_{L_{1,-1}}(S) = 0$, then for $\alpha \geqslant \beta$, we have

$$\mathcal{B}_\alpha v_0 = \big(\mathrm{ad}^\alpha_{L_{1,-1}}(\mathcal{B}_0)\big)v_0 = L^\alpha_{1,-1}\mathcal{B}_0 v_0 = L^\alpha_{1,-1} S v_0 = \big(\mathrm{ad}^\alpha_{L_{1,-1}}(S)\big)v_0 = 0.$$

This means that $\mathcal{B}_{[\beta,\infty)}v_0 = 0$, where $\mathcal{B}_{[\beta,\infty)} = \bigoplus_{\alpha \geqslant \beta} \mathcal{B}_\alpha$. The rest of the proof is exactly similar to that of [15, Proposition 2.1]. □

Lemma 2.3. *An irreducible uniformly bounded $\mathcal{B}$-module V is trivial.*

Proof. It is well known that a nontrivial highest or lowest weight module is not uniformly bounded (this can be also proved by regarding V as a $\mathrm{Vir}(\mathbb{Z})$-module, cf. [16]). Suppose V is nontrivial, then by Lemma 2.2, $L_{\pm 1,-1}$ acts nondegenerately on V. Thus there exists $N > 0$ such that $\dim V_\alpha = N$ for all $\alpha \in \mathbb{Z}$.

Claim 1. $L_{-1,1}|_V$ is bijective.

Denote $\mathfrak{g} = \mathrm{span}\{L_{-1,1}, L_{0,0}, L_{1,-1}\}$, the $sl(2)$-subalgebra of $\mathcal{B}$. Then V is a uniformly bounded $\mathfrak{g}$-module. Hence V has finite length as a $\mathfrak{g}$-module (cf. [12, Lemma 3.3]).

Assume that the action of $L_{-1,1}$ on V is not bijective. Then the action of $L_{-1,1}$ on V is neither injective nor surjective, in particular, there are some elements, annihilated by $L_{-1,1}$. Since V has a finite length as a $\mathfrak{g}$-module, standard $sl(2)$-theory implies that the number of weight spaces, where there are elements, annihilated by $L_{-1,1}$, is finite (as in any irreducible weight $sl(2)$-module the kernel of $L_{-1,1}$ is at most one-dimensional).

Hence we can fix the minimal such an element $\lambda \in \Gamma$ and some $0 \neq v \in V_\lambda$ such that $L_{-1,1}v = 0$. In particular,

$$L_{-1,1}|_{V_\mu} \text{ is injective} \quad \text{for } \mu < \lambda. \tag{2.1}$$

Set $T = \text{span}\{L_{\alpha,i} \mid L_{\alpha,i}v = 0\}$. We have $L_{-1,1} \in T$ and $\text{ad}_{L_{-1,1}} : T \to T$ (in fact T is certainly closed under the adjoint operation with respect to any element from T).

Subclaim (a). If $L_{\alpha,i} \in T$ and $\alpha < -1$, then $L_{\alpha+1,i-1} \in T$.

This follows from (2.1) and the facts that $L_{\alpha+1,i-1}v$ has degree $< \lambda$ and that

$$L_{-1,1}L_{\alpha+1,i-1}v = L_{\alpha+1,i-1}L_{-1,1}v + \big(2(\alpha+1)+i\big)L_{\alpha,i}v = 0 \quad \big(\text{cf. (1.1)}\big).$$

Let $K > 1$ be a positive integer such that for $i > K$ the restriction $L_{0,i}|_{V_\lambda}$ is a linear combination of $C|_{V_\lambda}$, $L_{0,j}|_{V_\lambda}$, $-1 \leqslant j \leqslant K$. Note that for $i \geqslant 0$ we have $\text{ad}_{L_{-1,1}}^{i+2}(L_{0,i}) = 0$.

Subclaim (b). $L_{-K-2,i} \in T$ for all $i > 2K+2$.

Set $\mathcal{C} = \bigoplus_{i>2K+2} \mathbb{F}L_{-K-2,i}$ and $\mathcal{D} = \bigoplus_{i>K} \mathbb{F}L_{0,i}$. Then $\mathcal{C} = \text{ad}_{L_{-1,1}}^{K+2}(\mathcal{D})$. We have

$$\begin{aligned}\mathcal{C}v &= \text{ad}_{L_{-1,1}}^{K+2}(\mathcal{D})v = L_{-1,1}^{K+2}\mathcal{D}v \subset L_{-1,1}^{K+2}\big(\text{span}\{L_{0,j}, C \mid j = -1, 0, \ldots, K\}\big)v \\ &= \text{ad}_{L_{-1,1}}^{K+2}\big(\text{span}\{L_{0,j}, C \mid j = -1, 0, \ldots, K\}\big)v = 0.\end{aligned}$$

Subclaim (c). $L_{\alpha,i} \in T$ for all $\alpha \leqslant -K-2$ and all i such that $2\alpha + i + 1 \geqslant 0$.

Applying $\text{ad}_{L_{-1,1}}$ to $\mathcal{C}$, we will get precisely the linear span of the elements in the formulation. The result now follows from the fact that T is stable under $\text{ad}_{L_{-1,1}}$.

For every negative integer β the elements $L_{\beta,i}v$, $i = -1, 0, \ldots, N$, are linearly dependent as the dimensions of all grading spaces of V are at most N. Hence for every such β there exists $i_0 = i_0(\beta) \in \{-1, 0, \ldots, N\}$ such that $\big(L_{\beta,i_0} + \sum_{i=i_0+1}^N a_{\beta,i}L_{\beta,i}\big)v = 0$ for some constants $a_{i,\beta}$. Set $u_\beta = L_{\beta,i_0} + \sum_{i=i_0+1}^N a_{\beta,i}L_{\beta,i}$. Certainly $[u_\beta, x]v = 0$ for every $x \in T$; in particular, $[u_\beta, \mathcal{C}]v = 0$.

For $\beta < -K-2$ and $i > 2(K+2-\beta)$ we have $2(-K-2+\beta) + (i-1) + 1 > 0$ and hence $L_{-K-2+\beta,i-1} \in T$ by Subclaim (c).

Subclaim (d). Let $\beta < -K-2-N$. Then $L_{-K-2+\beta,i} \in T$ for all $i > 2K+2+N$ (notice that the last estimate does not depend on β).

From the previous paragraph we know that the statement is true in the case $i > 2(K+2-\beta)$. Let us now show that if $i > 2K+2+N$, then we can do the downward induction, i.e., that $L_{-K-2+\beta,j} \in T$ for all $j > i$ implies that $L_{-K-2+\beta,i} \in T$. This will certainly do the job.

So, let $i > 2K + 2 + N$ and assume that $L_{-K-2+\beta,j} \in T$ for all $j > i$. Consider the element $L_{-K-2,i-i_0}$. Since $i_0 \leqslant N$ and $i > 2K + 2 + N$, we have $i - i_0 > 2K + 2$ and hence $L_{-K-2,i-i_0} \in T$ by Subclaim (b). Furthermore, we have

$$0 = [u_\beta, L_{-K-2,i-i_0}]v = \left(bL_{-K-2+\beta,i} + \sum_{s=i_0+1}^{N} b_{\beta,s}a_{\beta,s}L_{-K-2+\beta,i-i_0+s}\right)v$$

for some constants $b, b_{\beta,s}$. Since $i - i_0 + s > i$, we have $L_{-K-2+\beta,i-i_0+s} \in T$ (by inductive assumption) and hence $bL_{-K-2+\beta,i}v = 0$. Note that for all $\beta < -K - 2 - N$ the point (β, i_0) (with $i_0 \leqslant N$) satisfies $2\beta + i_0 + 1 < 0$ and hence does not belong to the straight line through $(0, -1)$ and $(-K - 2, i - i_0)$ (all points (x, y) with $x < 0$ on the latter line obviously satisfy $2x + y + 1 > 0$). This implies that L_{β,i_0} and $L_{-K-2,i-i_0}$ do not commute and hence $b \neq 0$. Thus $L_{-K-2+\beta,i}v = 0$ and $L_{-K-2+\beta,i} \in T$.

Take now all $\beta < -(3K + 2N + 6)$ and apply $2K + N + 4$ times Subclaim (a) to the subset of T given by Subclaim (d). It follows that $L_{\beta,i} \in T$ for all $\beta < -(K + N + 2)$ and all $i \geqslant -1$. Now applying Subclaim (a) to the last obtained set $K + N + 2$ times, we obtain that $L_{\beta,i} \in T$ for all $i \geqslant -1$ and all negative β. Hence V is a lowest weight module. A contradiction. This completes the proof of Claim 1.

Now note that the element $L_{1,-1}$ is locally nilpotent on $\mathcal{B}$, hence we can form the Ore localization of $\mathcal{U}(\mathcal{B})$ (the universal enveloping algebra of $\mathcal{B}$) with respect to $\{L_{1,-1}^i \mid i \in \mathbb{N}\}$ and consider the corresponding Mathieu automorphisms of the localized algebra (see [12, Lemmas 4.2 and 4.3]). Now, twisting the module V with respect to these automorphisms, we will get many new irreducible quasifinite uniformly bounded $\mathcal{B}$-modules. But, using [12, Lemma 5.1], it follows that in some of these modules the action of the element $L_{-1,1}$ will not be bijective. Using now the first statement of the proof of the lemma (that there are no nontrivial highest weight uniformly bounded $\mathcal{B}$-modules), we get a contradiction with Claim 1. This completes the proof of the lemma. □

Theorem 2.4. *A quasifinite irreducible $\mathcal{B}(\Gamma)$-module V is trivial if $\Gamma \ncong \mathbb{Z}$.*

Proof. Regard V as a $\mathrm{Vir}(\Gamma)$-module and use the results in [16] (cf. [15, Proposition 3.1]), We can prove: if V is nontrivial, then there exists a (rank one) centerless Virasoro subalgebra $\mathrm{Vir}(\mathbb{Z}a)$ (for some $a \in \Gamma$) and there exists a nontrivial composition factor W of V (regarding as a $\mathrm{Vir}(\mathbb{Z}a)$-module), such that W is a module of the intermediate series; and furthermore, we can obtain that there exists a nontrivial uniformly bounded module over $\mathcal{B}(\mathbb{Z}a)$, a contradiction with Lemma 2.3. □

3. Quasifinite irreducible highest weight modules over $\mathcal{B}$

Now we consider the Lie algebra $\mathcal{B} = \mathcal{B}(\mathbb{Z})$. We follow [7] closely in this section. A *Verma module* over $\mathcal{B}$ is defined as the induced module

$$M(\Lambda) = \mathcal{U}(\mathcal{B}) \otimes_{\mathcal{U}(\mathcal{B}_0 \oplus \mathcal{B}_+)} \mathbb{F}_\Lambda,$$

where $\mathbb{F}_\Lambda$ is the one-dimensional $(\mathcal{B}_0 \oplus \mathcal{B}_+)$-module given by $(h+n)(c) = \Lambda(h)c$ for $h \in \mathcal{B}_0$, $n \in \mathcal{B}_+$, and $c \in \mathbb{F}_\Lambda$, and where in general, $\mathcal{U}(\mathcal{L})$ stands for the universal enveloping algebra of a Lie algebra $\mathcal{L}$. Then any highest weight module $V(\Lambda)$ is a quotient module of $M(\Lambda)$ and the irreducible highest weight module $L(\Lambda)$ is the quotient of $M(\Lambda)$ by the maximal proper $\mathbb{Z}$-graded submodule.

Suppose $\mathcal{P}$ is a *parabolic subalgebra* of $\mathcal{B}$, i.e.,

$$\mathcal{P} \supset \mathcal{B}_0 \oplus \mathcal{B}_+ \neq \mathcal{P}.$$

Then $\mathcal{P} = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{P}_\alpha$ such that $\mathcal{P}_\alpha = \mathcal{B}_\alpha$ for $\alpha \geqslant 0$ and $P_\alpha \neq \{0\}$ for some $\alpha < 0$. Let $\Lambda \in \mathcal{B}_0^*$ be such that $\Lambda|_{\mathcal{B}_0 \cap [\mathcal{P},\mathcal{P}]} = 0$. Then the $(\mathcal{B}_0 \oplus \mathcal{B}_+)$-module $\mathbb{F}_\Lambda$ extends to a $\mathcal{P}$-module by letting $\mathcal{P}_\alpha$ act as zero for $\alpha < 0$. We define the highest weight module

$$M(\mathcal{P}, \Lambda) = \mathcal{U}(\mathcal{B}) \otimes_{\mathcal{U}(\mathcal{P})} \mathbb{F}_\Lambda,$$

called a *generalized Verma module*.

Given nonzero $a \in \mathcal{B}_{-1}$, we define

$$\mathcal{P}^a = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{P}^a_\alpha,$$

to be the *minimal parabolic subalgebra containing a*. Then by [7], $\mathcal{P}^a_\alpha = \mathcal{B}_\alpha$ for $\alpha \geqslant 0$ and

$$\mathcal{P}^a_{-1} = \operatorname{span}\bigl\{\bigl[\ldots\bigl[[a, \mathcal{B}_0], \mathcal{B}_0\bigr], \ldots,\bigr]\bigr\}, \qquad \mathcal{P}^a_{-\alpha-1} = \bigl[\mathcal{P}^a_{-1}, \mathcal{P}^a_{-\alpha}\bigr].$$

Also, we have

$$\mathcal{B}^a_0 := \bigl[\mathcal{P}^a, \mathcal{P}^a\bigr] \cap \mathcal{B}_0 = [a, \mathcal{B}_1]. \tag{3.1}$$

A parabolic subalgebra $\mathcal{P}$ is *nondegenerate* if $\mathcal{P}_{-\alpha}$ has finite codimension in $\mathcal{B}_{-\alpha}$ for all $\alpha > 0$; a nonzero element $a \in \mathcal{B}_{-1}$ is *nondegenerate* if $\mathcal{P}^a$ is nondegenerate. We have the following lemma.

Lemma 3.1. *The Lie algebra $\mathcal{B}$ satisfies the following three properties*:

(i) *$\mathcal{B}_0$ is commutative*;
(ii) *if $a \in \mathcal{B}_{-\beta}$ for some $\beta > 0$ and $[a, \mathcal{B}_1] = 0$, then $a = 0$*;
(iii) *if $\mathcal{P}$ is a nondegenerate parabolic subalgebra of $\mathcal{B}$, then there exists a nondegenerate element a such that $\mathcal{P}^a \subset \mathcal{P}$.*

Proof. (i) and (ii) follow directly from (1.1), while (iii) follows from Lemma 3.2 below. □

Lemma 3.2.

(1) *Any nonzero element $a \in \mathcal{B}_{-1}$ is nondegenerate.*

(2) *Any parabolic subalgebra of $\mathcal{B}$ is nondegenerate.*
(3) *Let $a = x^{-1} f(t) \in \mathcal{B}_{-1}$, then*

$$\mathcal{B}_0^a = \operatorname{span}\bigl\{\bigl(f(t)g(t)\bigr)' - f(0)g(0)C \bigm| g(t) \in \mathbb{F}[t]\bigr\}.$$

Proof. Suppose $\mathcal{P}$ is a parabolic subalgebra. Then $\mathcal{P}_{-\beta} \neq \{0\}$ for some $\beta > 0$. Let

$$I_{-\beta} = \bigl\{f(t) \in \mathbb{F}[t] \bigm| x^{-\beta} f(t) \in \mathcal{P}_{-\beta}\bigr\}.$$

Since $[x^{-\beta} f(t), g(t)] = \beta x^{-\beta} f(t) g'(t)$ for all $g(t) \in \mathbb{F}[t]$, we obtain $f(t)\mathbb{F}[t] \subset I_{-\beta}$ if $f(t) \in I_{-\beta}$. In particular, $\mathcal{P}_{-\beta}$ has finite codimension in $\mathcal{B}_{-\beta}$. Using (1.1), by induction on β, we have $\mathcal{P}_{-1} \neq \{0\}$. Then by induction on $\alpha > 0$, we obtain $\mathcal{P}_{-\alpha} \neq \{0\}$, and so $\mathcal{P}_{-\alpha}$ has finite codimension in $\mathcal{B}_{-\alpha}$. Thus we have (2), which implies (1). By (1.1),

$$\bigl[x^{-1} f(t), x g(t)\bigr] = f'(t)g(t) + f(t)g'(t) - f(0)g(0)C,$$

for $g(t) \in \mathbb{F}[t]$, so (3) follows from (3.1). $\square$

By Lemma 3.1 and [7, Theorem 2.5], we have the following lemma.

Lemma 3.3. *The following conditions on $\Lambda \in \mathcal{B}_0^*$ are equivalent*:

(1) *$M(\Lambda)$ contains a singular vector $a \cdot v_\Lambda$ in $M(\Lambda)_{-1}$, where a is nondegenerate*;
(2) *there exists a nondegenerate element $a \in \mathcal{B}_{-1}$ such that $\Lambda(\mathcal{B}_0^a) = 0$*;
(3) *$L(\Lambda)$ is quasifinite*;
(4) *there exists a nondegenerate element $a \in \mathcal{B}_{-1}$ such that $L(\Lambda)$ is the irreducible quotient of the generalized Verma module $M(\mathcal{P}^a, \Lambda)$.*

Let $L(\Lambda)$ be a quasifinite irreducible highest weight module over $\mathcal{B}$. By Lemma 3.3, there exists some monic polynomial $f(t)$ such that $(x^{-1} f(t))v_\Lambda = 0$. We shall call such monic polynomial of minimal degree, uniquely determined by the highest weight Λ, the *characteristic polynomial* of $L(\Lambda)$.

A function $\Lambda \in \mathcal{B}_0^*$ is described by the *central charge* $c = \Lambda(C)$ and its *labels* $\Lambda_i = -\Lambda(t^i)$ for $i \geqslant 0$. We introduce the *generating series*

$$\Delta_\Lambda(z) = c + \sum_{i=0}^{\infty} \frac{z^{i+1}}{i!} \Lambda_i = c - \Lambda\bigl(z e^{zt}\bigr). \tag{3.2}$$

A *quasipolynomial* $\Delta(z)$ is a linear combination of functions of the form $p(z)e^{\alpha z}$, where $p(z) \in \mathbb{F}[z]$, $\alpha \in \mathbb{F}$. Recall [7,8] the well-known characterization that a formal power series is a quasipolynomial if and only if it satisfies a nontrivial linear differential equation with constant coefficients.

We have the following characterization of quasifinite irreducible highest weight modules.

Theorem 3.4. *A module $L(\Lambda)$ over $\mathcal{B}$ is quasifinite if and only if $\Delta_\Lambda(z)$ is a quasipolynomial.*

Proof. By Lemmas 3.2(3) and 3.3(2), we see that $L(\Lambda)$ is quasifinite if and only if there exists a polynomial $f(t)$ such that

$$\Lambda\big(\big(f(t)g(t)\big)' - f(0)g(0)C\big) = 0 \quad \text{for all } g(t) \in \mathbb{F}[t].$$

Equivalently, using $e^{zt} = \sum_{i\geqslant 0}(z^i/i!)t^i$ as a generating series of $\mathbb{F}[t]$, we can take $g(t) = e^{zt}$; then by $f(t)e^{zt} = f(\partial/\partial z)e^{zt}$ and by recalling that the prime stands for $\partial/\partial t$, we have

$$\begin{aligned} 0 &= -\Lambda\big(\big(f(t)e^{zt}\big)' - f(0)C\big) = -\Lambda\left(\left(f\left(\frac{\partial}{\partial z}\right)e^{zt}\right)'\right) + f(0)c \\ &= -\Lambda\left(f\left(\frac{\partial}{\partial z}\right)ze^{zt}\right) + f(0)c = -f\left(\frac{\partial}{\partial z}\right)\Lambda\big(ze^{zt}\big) + f(0)c \\ &= f\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\big(\Delta_\Lambda(z) - c\big) + f(0)c = f\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\big(\Delta_\Lambda(z)\big), \end{aligned} \tag{3.3}$$

i.e., $L(\Lambda)$ is quasifinite if and only if $\Delta_\Lambda(z)$ is a quasipolynomial. □

By definition, a quasipolynomial $\Delta_\Lambda(z)$ in Theorem 3.4 can be uniquely written in the form

$$\Delta_\Lambda(z) = \sum_{\gamma\in I} p_\gamma(z)e^{\gamma z}, \tag{3.4}$$

where $I \subset \mathbb{F}$ is a finite subset such that $p_\gamma(z) \neq 0$ if $\gamma \in I$. The numbers $\gamma \in I$ are *exponents* of the $\mathcal{B}$-module $L(\Lambda)$, and $p_\gamma(z)$ is the *multiplicity* of γ, denoted by $\mathrm{mult}(\gamma)$. Clearly, the central charge is

$$c = \sum_{\gamma\in I} p_\gamma(0).$$

The following is clear by the proof of Theorem 3.4.

Corollary 3.5. *Let $L(\Lambda)$ be a quasifinite irreducible highest weight module over $\mathcal{B}$ such that $f(t)$ is the characteristic polynomial. Then $f(\mathrm{d}/\mathrm{d}z)\Delta_\Lambda(z) = 0$ is the minimal order homogeneous linear differential equation with constant coefficients satisfied by $\Delta_\Lambda(z)$. Furthermore, the exponents $\gamma \in I$ are all roots of the polynomial of $f(t)$.*

4. Unitary quasifinite irreducible highest weight modules over $\mathcal{B}$ are trivial

In this section, we shall describe unitary irreducible highest weight modules. Unfortunately, unlike $W_{1+\infty}$, by Theorem 2.1 one can see that there is no natural embedding of $\mathcal{B}$ into $\widehat{gl}_\infty$. We shall prove that a unitary quasifinite irreducible highest weight module over $\mathcal{B}$ is trivial.

Now we assume that $\mathbb{F} = \mathbb{C}$. Consider the $\mathbb{R}$-linear map $\omega : \mathcal{B} \to \mathcal{B}$ defined by

$$\omega\big(x^\alpha f(t)\big) = x^{-\alpha}\overline{f}(t), \tag{4.1}$$

where $\overline{f}(t) = \sum_{i\geqslant 0}\overline{f}_i t^i$ for $f(t) = \sum_{i\geqslant 0} f_i t^i$, $f_i \in \mathbb{C}$, and $\overline{f}_i$ is the conjugate number of f_i.

Lemma 4.1. *The map ω is an anti-involution of $\mathcal{B}$, i.e.,*

$$\omega^2 = \mathrm{id}, \qquad \omega(\lambda a) = \bar{\lambda}\omega(a), \qquad \omega\big([a,b]\big) = \big[\omega(b), \omega(a)\big], \tag{4.2}$$

for $\lambda \in \mathbb{C}$, $a, b \in \mathcal{B}$. Furthermore, $\omega(\mathcal{B}_\alpha) = \mathcal{B}_{-\alpha}$ for $\alpha \in \mathbb{Z}$.

Proof. First two equations of (4.2) are obvious. We have

$$\begin{aligned}
\omega\big(\big[x^\alpha f(t), x^\beta g(t)\big]\big) &= \omega\big(x^{\alpha+\beta}\big(\beta f'(t)g(t) - \alpha f(t)g'(t)\big)\big)\\
&= x^{-\alpha-\beta}\big(\beta\overline{f}'(t)\overline{g}(t) - \alpha\overline{f}(t)\overline{g}'(t)\big)\\
&= \big[x^{-\beta}\overline{g}(t), x^{-\alpha}\overline{f}(t)\big] = \big[\omega\big(x^\beta g(t)\big), \omega\big(x^\alpha f(t)\big)\big].
\end{aligned}$$

Thus ω is an anti-involution. □

A module V over $\mathcal{B}$ is *unitary* (with respect to the anti-involution ω) if there exists a positive definite Hermitian form $H(\cdot\,,\cdot)$ on V such that it is *contravariant* (i.e., $\omega(a)$ and a are adjoint operators on V with respect to H for $a \in \mathcal{B}$).

Lemma 4.2. *Suppose $V = L(\Lambda)$ is a unitary quasifinite irreducible highest weight $\mathcal{B}$-module and $f(t)$ is the characteristic polynomial. Then $f(t)$ has only simple real roots. In particular, $\Delta_\Lambda(z) = \sum_{\gamma\in I} p_\gamma \mathrm{e}^{\gamma z}$ for some $I \subset \mathbb{R}$ and $p_\gamma \in \mathbb{C}$.*

Proof. First, using $H(t^i v_\Lambda, v_\Lambda) = H(v_\Lambda, t^i v_\Lambda)$, we obtain that Λ_i are real for $i \geqslant 0$. Similarly, the central charge c is also real. From definition of $f(t)$, we obtain that V_{-1} has a basis

$$\big\{\big(x^{-1}t^i\big)v_\Lambda \;\big|\; 0 \leqslant i < \deg f\big\}.$$

Let $T = -\frac{1}{2}(t^2 + \Lambda_2) \in \mathcal{B}_0$. By induction on $i \geqslant 0$, we have $T^i(x^{-1}v_\Lambda) = (x^{-1}t^i)v_\Lambda$. It follows that $f(T)(x^{-1}v_\Lambda) = 0$ and that $\{T^i(x^{-1}v_\Lambda) \mid 0 \leqslant i < \deg f\}$ is a basis of V_{-1}.

We obtain that $f(t)$ is the characteristic polynomial of the operator $T|_{V_{-1}}$. Since $T|_{V_{-1}}$ is self-adjoint, all the roots of $f(t)$ are real.

Suppose $f(t)=(t-\gamma)^m g(t)$ with $\gamma\in\mathbb{R}$, $m\geqslant 1$ and $g(t)\in\mathbb{C}[t]$. Then

$$v=(T-\gamma)^{m-1}g(T)\big(x^{-1}v_\Lambda\big)\in V_{-1},$$

is nonzero, but

$$H(v,v)=H\big(g(T)\big(x^{-1}v_\Lambda\big),(T-\gamma)^{2m-2}g(T)\big(x^{-1}v_\Lambda\big)\big)=0\quad\text{if } m\geqslant 2.$$

Hence the unitarity condition implies $m=1$. The last statement of Lemma 4.2 follows from (3.4) and Corollary 3.5. □

Theorem 4.3. *If $L(\Lambda)$ is unitary, then it is trivial.*

Proof. Suppose $L(\Lambda)$ is nontrivial. Then the characteristic polynomial $f(t)$ is not a constant. Let $u=(x^{-2}f(t))v_\Lambda\in L(\Lambda)_{-2}$. Then

$$x^1\cdot u=\big[x^1,x^{-2}f(t)\big]v_\Lambda=-\big(x^{-1}f'(t)\big)v_\Lambda\neq 0.$$

Thus $u\neq 0$. Assume that $H(v_\Lambda,v_\Lambda)=1$. Then by (1.2), (3.2), (3.3), (4.1), and the facts that $f(t)$ is a real polynomial and that $f(t)=f(\partial/\partial z)\mathrm{e}^{zt}|_{z=0}$, we have

$$\begin{aligned}
0<H(u,u)&=H\big(\big(x^2f(t)\big)\big(x^{-2}f(t)\big)v_\Lambda,v_\Lambda\big)=2H\big(\big(f(0)^2C-\big(f(t)^2\big)'\big)v_\Lambda,v_\Lambda\big)\\
&=2f(0)^2c-2H\left(\left(f\left(\frac{\partial}{\partial z}\right)^2\mathrm{e}^{zt}\right)'v_\Lambda,v_\Lambda\right)\bigg|_{z=0}\\
&=2f(0)^2c-2f\left(\frac{\partial}{\partial z}\right)^2H\big(z\mathrm{e}^{zt}v_\Lambda,v_\Lambda\big)\big|_{z=0}\\
&=2f\left(\frac{d}{dz}\right)^2\Delta_\Lambda(z)|_{z=0}\\
&=0,
\end{aligned}$$

a contradiction, where the last equality follows from Corollary 3.5. □

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References

[1] R. Block, On torsion-free abelian groups and Lie algebras, Proc. Amer. Math. Soc. 9 (1958) 613–620.
[2] P. Boyallin, V. Kac, J. Liberati, C. Yan, Quasifinite highest weight modules of the Lie algebra of matrix differential operators on the circle, J. Math. Phys. 39 (1998) 2910–2928.
[3] A. Capell, C.A. Trugenberger, G.R. Zemba, Classifications of quantum Hall universality classes by $W_{1+\infty}$ symmetry, Phys. Rev. Lett. 72 (1994) 1902–1905.
[4] V. Chari, Integrable representations of affine Lie algebras, Invent. Math. 85 (1986) 317–335.
[5] D. Dokovic, K. Zhao, Derivations, isomorphisms and second cohomology of generalized Block algebras, Algebra Colloq. 3 (1996) 245–272.
[6] E. Frenkel, V. Kac, R. Radul, W. Wang, $W_{1+\infty}$ and $W(gl_N)$ with central charge N, Comm. Math. Phys. 170 (1995) 337–357.
[7] V. Kac, J. Liberati, Unitary quasi-finite representations of W_∞, Lett. Math. Phys. 53 (2000) 11–27.
[8] V. Kac, A. Radul, Quasi-finite highest weight modules over the Lie algebra of differential operators on the circle, Comm. Math. Phys. 157 (1993) 429–457.
[9] V. Kac, A. Radul, Representation theory of the vertex algebra $W_{1+\infty}$, Transform. Groups 1 (1996) 41–70.
[10] V. Kac, W. Wang, C. Yan, Quasifinite representations of classical Lie subalgebras of $W_{1+\infty}$, Adv. Math. 139 (1998) 46–140.
[11] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro Lie algebra, Invent. Math. 107 (1992) 225–234.
[12] O. Mathieu, Classification of irreducible weight modules, Ann. Inst. Fourier (Grenoble) 50 (2) (2000) 537–592.
[13] J.M. Osborn, K. Zhao, Infinite-dimensional Lie algebras of generalized Block type, Proc. Amer. Math. Soc. 127 (1999) 1641–1650.
[14] Y. Su, Derivations and structure of the Lie algebras of Xu type, Manuscripta Math. 105 (2001) 483–500.
[15] Y. Su, Classification of quasifinite modules over the Lie algebras of Weyl type, Adv. Math. 174 (2003) 57–68.
[16] Y. Su, Classification of Harish-Chandra modules over the higher rank Virasoro algebras, Comm. Math. Phys. 240 (2003) 539–551.
[17] Y. Su, Structure of Lie superalgebras of Block type related to locally finite derivations, Comm. Algebra 31 (2003) 1725–1751.
[18] Y. Su, J. Zhou, Structure of the Lie algebras with a feature of Block algebras, Comm. Algebra 30 (2002) 3205–3226.
[19] X. Xu, Generalizations of Block algebras, Manuscripta Math. 100 (1999) 489–518.
[20] X. Xu, Quadratic conformal superalgebras, J. Algebra 224 (2000) 1–38.
[21] H. Zhang, K. Zhao, Representations of the Virasoro-like Lie algebra and its q-analog, Comm. Algebra 24 (1996) 4361–4372.
[22] K. Zhao, A class of infinite dimensional simple Lie algebras, J. London Math. Soc. (2) 62 (2000) 71–84.
[23] L. Zhu, D. Meng, Structure of degenerate Block algebras, Algebra Colloq. 10 (2003) 53–62.